Quantum Cuntz-Krieger algebras

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Overview

The aim of my talk is to

- describe quantum graphs, a notion generalising finite graphs within the framework of operator algebras and noncommutative geometry,
- explain why one may wish to study these objects,
- explain how one can associate operator algebras to quantum graphs, in analogy to the construction of Cuntz-Krieger algebras, and
- discuss some examples.

Graphs

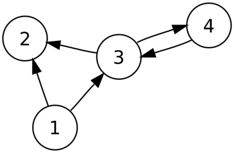
Definition

A (finite, directed) graph $E = (E^0, E^1, s, r)$ is given by

- finite sets E^0, E^1 of *vertices* and *edges*, respectively,
- maps $s, r : E^1 \to E^0$, called *source* and *range*.

Given $e \in E^1$ we say that e is an edge from $s(e)$ to $r(e)$.

We will only consider *simple* graphs, that is, graphs for which there is at most one edge from v to w for all vertices $v, w \in E^0$.



Graphs

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The *adjacency matrix* of a (simple) graph $E = (E^0, E^1, s, r)$ is the matrix $A = A_E \in M_{E^0}(\{0,1\})$ determined by

$$A(v,w) = 1 \Leftrightarrow (v,w) \in E^1.$$

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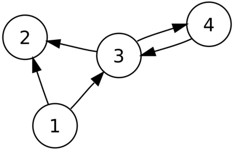
Consider again the example from the previous slide:



The adjacency matrix of this graph is

$$A_E = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Towards quantum graphs

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To this end we

- replace the set E^0 of vertices with the finite dimensional commutative C^*-algebra $B = C(E^0)$, spanned linearly by the pairwise orthogonal projections p_v for $v \in E^0$.
- equip B with the "canonical" tracial state $\psi : B \to \mathbb{C}$ given by

$$\psi(x) = \mathrm{tr}_{\mathrm{End}(B)}(x),$$

  where $B \subset \mathrm{End}(B)$ via (left) multiplication.
- use the GNS-construction of ψ to view $B = L^2(B)$ as a (finite dimensional) Hilbert space.
- view the adjacency matrix A_E as a linear operator $A : L^2(B) \to L^2(B)$ via

$$A(p_v) = \sum_{w \in E^0} A(v, w) p_w.$$

Towards quantum graphs

Question

How to capture the fact that $A : L^2(B) \to L^2(B)$ is the adjacency matrix of E in terms of $B = C(E^0)$ and ψ?

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$A \in M_{E^0}(\mathbb{C}) = B(L^2(B))$ is the adjacency matrix of a (uniquely determined) graph with vertex set E^0 iff the following equivalent conditions hold:

- A has entries in $0, 1$
- A is an idempotent with respect to Schur product, that is, $A \star A = A$.

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- A is an idempotent with respect to Schur product, that is, $A \star A = A$.

Let $m : L^2(B) \otimes L^2(B) \to L^2(B)$ be the multiplication map and m^* its Hilbert space adjoint. Explicitly,

$$m(p_v \otimes p_w) = \delta_{v,w} p_v, \qquad m^*(p_v) = \dim(B) p_v \otimes p_v.$$

One calculates

$$X \star Y = \frac{1}{\dim(B)} m(X \otimes Y) m^*$$

for the *Schur product* of matrices $X, Y \in M_{E^0}(\mathbb{C})$.

Conclusion

A is the adjacency matrix of a graph iff $m(A \otimes A)m^* = \dim(B)A$.

Quantum graphs

Quantum graphs

The following definition goes back to work of Musto-Reutter-Verdon (2018) and Brannan-Chirvasitu-Eifler-Harris-Paulsen-Su-Wasilewski (2019).

Definition

A (tracial, directed) quantum graph $\mathcal{G} = (B, \psi, A)$ consists of

- a finite dimensional C^*-algebra B,
- the canonical tracial state $\psi : B \to \mathbb{C}$ obtained from the normalised trace on $\mathrm{End}(B)$ via $B \subset \mathrm{End}(B)$,
- a linear operator $A : L^2(B) \to L^2(B)$, called quantum adjacency matrix, satisfying

$$m(A \otimes A)m^* = \dim(B)A.$$

Slogan

We are replacing the vertices of a graph by finite dimensional matrix algebras.

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Lemma

A quantum graph (B, ψ, A) with B commutative is the same thing as a (finite, directed, simple) graph.

Some examples

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Consider an arbitrary finite dimensional C^*-algebra B.

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Example

- The *complete quantum graph* on B is (B, ψ, A) with

$$A(x) = \dim(B)\psi(x)1$$

  Classically,

$$\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

- The *trivial quantum graph* on B is (B, ψ, A) with $A = \mathrm{id} : B \to B$.

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- The *trivial quantum graph* on B is (B, ψ, A) with $A = \mathrm{id} : B \to B$.
- If $B = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})$ and $d = \bigoplus_{i=1}^{N} d_i \in B$ a direct sum of diagonal matrices, then

$$A(x) = dx$$

  defines a quantum graph iff $Tr(d_i) = n_i$ for all i.

Quantum graphs - alternative picture

Quantum graphs - alternative picture

A (simple) graph $\mathcal{G} = (E^0, E^1, r, s)$ is completely determined by the edge relation $R \subset E^0 \times E^0$, where $(v, w) \in R$ iff $v = s(e), w = r(e)$ for $e \in E^1$.

Consider the subspace $S_\mathcal{G} \subset B(l^2(E^0))$ spanned by all rank-one operators $e_{v,w}$ with $(v, w) \in E^1$. We may think of $e_{v,w}$ as the "edge operator" associated with (v, w).

Definition (Weaver 2010)

Let H be a finite dimensional Hilbert space and let $B \subset B(H)$ be a C^*-algebra. A quantum graph on B is a B'-B'-bimodule $S \subset B(H)$.

The bimodule contains the "edge operators" connecting "quantum vertices" in the graph.

Comparison between the two approaches

Given a finite dimensional Hilbert space H and a C^*-algebra $B \subset B(H)$, an idempotent $P = \sum a_i \otimes b_i^{opp} \in B \otimes B^{opp}$ determines a B'-B'-bimodule $S \subset B(H)$ via

$$S = P \cdot B(H) = \left\{ \sum a_i X b_i \mid X \in B(H) \right\}.$$

Here B' is the commutant of B.

Lemma

Any idempotent $P \in B \otimes B^{opp}$ arises as Choi-Jamoiłkowski matrix of a quantum adjacency matrix A via

$$P = \frac{1}{\dim(B)} (A \otimes 1) m^*(1).$$

Conclusion

Quantum adjacency matrices are the same thing as direct sum decompositions $B(H) = S \oplus R$ of B'-B'-bimodules.

In particular, there are many quantum graphs not coming from classical graphs.

Quantum graphs “in nature”

Quantum graphs appear in

- the graph isomorphism game and the study of quantum symmetry groups of graphs
- quantum teleportation and superdense coding schemes
- the definition of zero-error capacity of quantum channels

...and probably more applications yet to be discovered.

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Write $L^2(B) = L^2(B, \psi)$ for the GNS-construction of ψ and let $m : L^2(B) \otimes L^2(B) \to L^2(B)$ be the multiplication map.

Definition

If $\delta > 0$ then $\psi : B \to \mathbb{C}$ is called a δ-form if $mm^* = \delta^2 \, \mathrm{id}$.

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If $\delta > 0$ then $\psi : B \to \mathbb{C}$ is called a δ-form if $mm^* = \delta^2 \,\mathrm{id}$.

Any finite dimensional C^*-algebra admits a unique *tracial* δ-form with $\delta^2 = \dim(B)$.

Definition

A (directed) quantum graph $\mathcal{G} = (B, \psi, A)$ consists of

- a finite dimensional C^*-algebra B,
- a δ-form $\psi : B \to \mathbb{C}$,
- a linear operator $A : L^2(B) \to L^2(B)$, called quantum adjacency matrix, satisfying

$$m(A \otimes A)m^* = \delta^2 A.$$

Cuntz-Krieger algebras

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Definition

Let $A \in M_N(\mathbb{Z})$ be a matrix with entries $A(i,j) \in \{0,1\}$. The Cuntz-Krieger algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by partial isometries $S_1, \ldots, S_N$ with mutually orthogonal ranges, satisfying

$$S_i^* S_i = \sum_{j=1}^{N} A(i,j) S_j S_j^*$$

for all $1 \leq i \leq N$.

Example

Let $A = \mathrm{id} \in M_N(\mathbb{Z})$. Then $\mathcal{O}_A = C(S^1) \oplus \cdots \oplus C(S^1)$ is the direct product of N copies of $C(S^1)$.

Example

If $A = (a_{ij}) \in M_N(\mathbb{Z})$ with $a_{ij} = 1$ for all i,j then $\mathcal{O}_A = \mathcal{O}_N$ is the Cuntz algebra.

Free Cuntz-Krieger algebras

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Definition

Let $A \in M_N(\mathbb{Z})$ be a matrix with entries $A(i,j) \in \{0,1\}$. The *free* Cuntz-Krieger algebra $\mathbb{F}\mathcal{O}_A$ is the the universal C^*-algebra generated by partial isometries $S_1, \ldots, S_N$, satisfying

$$S_i^* S_i = \sum_{j=1}^{N} A(i,j) S_j S_j^*$$

for all $1 \leq i \leq N$.

The only difference is that we do *not* require the partial isometries S_i to have mutually orthogonal ranges.

Example

Let $A = \mathrm{id} \in M_N(\mathbb{Z})$. Then $\mathbb{F}\mathcal{O}_A = C(S^1) * \cdots * C(S^1)$ is the non-unital free product of N copies of $C(S^1)$.

Example

If $A = (a_{ij}) \in M_N(\mathbb{Z})$ with $a_{ij} = 1$ for all i,j then $\mathbb{F}\mathcal{O}_A = \mathcal{O}_N$ is (still) the Cuntz algebra.

Free Cuntz-Krieger algebras

Question

What is the relation between free Cuntz-Krieger algebras and Cuntz-Krieger algebras in general?

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Theorem

Let $A \in M_N(\mathbb{Z})$ be a matrix with entries $A(i,j) \in \{0,1\}$. The canonical quotient map

$$\mathbb{F}\mathcal{O}_A \to \mathcal{O}_A$$

is a KK-equivalence.

The proof is an adaption of a well-known argument due to Cuntz showing that a non-unital free product $A * B$ is KK-equivalent to $A \oplus B$.

Quantum Cuntz-Krieger algebras

Quantum Cuntz-Krieger algebras

Let $\mathcal{G} = (B, \psi, A)$ be a directed quantum graph.

We shall say that a quantum Cuntz-Krieger $\mathcal{G}$-family in a C^*-algebra D is a linear map $s : B \to D$ such that

a) $\mu_D(\mathrm{id} \otimes \mu_D)(s \otimes s^* \otimes s)(\mathrm{id} \otimes m^*)m^* = s$

b) $\mu_D(s^* \otimes s)m^* = \mu_D(s \otimes s^*)m^* A$.

Here $\mu_D : D \otimes D \to D$ is the multiplication map for D and $s^*(b) = s(b^*)^*$ for $b \in B$.

Definition

Let $\mathcal{G} = (B, \psi, A)$ be a directed quantum graph. The quantum Cuntz-Krieger algebra $\mathbb{FO}(\mathcal{G})$ is the universal C^*-algebra generated by a quantum Cuntz-Krieger $\mathcal{G}$-family $S : B \to \mathbb{FO}(\mathcal{G})$.

Example: Classical graphs

Let $E = (E^0, E^1, r, s)$ be a graph with N vertices.

The associated quantum graph $\mathcal{G} = (B, \psi, A)$ has $B = C(E^0) = \mathbb{C}^N$ as underlying C^*-algebra, equipped with the canonical trace ψ.
If A_E denotes the adjacency matrix of E then

$$A(e_i) = \sum_{j=1}^{N} A_E(i,j) e_j$$

determines a quantum adjacency matrix $A : L^2(B) \to L^2(B)$.

Proposition

Let $\mathcal{G} = (B, \psi, A)$ be the quantum graph corresponding to the classical graph E as above. Then the free Cuntz-Krieger algebra associated with the adjacency matrix A_E is canonically isomorphic to the quantum Cuntz-Krieger algebra $\mathbb{F}\mathcal{O}(\mathcal{G})$.

Example: Classical graphs

Proof.

Consider $S_i = NS(e_i) \in \mathbb{F}\mathcal{O}(\mathcal{G})$. Then

$$\begin{aligned} S_i S_i^* S_i &= N^3 \mu(\mathrm{id} \otimes \mu)(S(e_i) \otimes S^*(e_i) \otimes S(e_i)) \\ &= N\mu(\mathrm{id} \otimes \mu)(S \otimes S^* \otimes S)(\mathrm{id} \otimes m^*) m^*(e_i) \\ &= NS(e_i) = S_i \end{aligned}$$

and

$$\begin{aligned} S_i^* S_i = N^2 \mu(S^* \otimes S)(e_i \otimes e_i) &= N\mu(S^* \otimes S) m^*(e_i) \\ &= N\mu(S \otimes S^*) m^*(A(e_i)) \\ &= N^2 \sum_{j=1}^{N} A_E(i,j) \mu(S \otimes S^*)(e_j \otimes e_j) \\ &= \sum_{j=1}^{N} A_E(i,j) S_j S_j^* \end{aligned}$$

for all i. This yields a $*$-isomorphism $\mathbb{F}\mathcal{O}_{A_E} \to \mathbb{F}\mathcal{O}(\mathcal{G})$. □

Example: Trivial quantum graphs

Let $B = M_N(\mathbb{C})$ with its normalised trace $\mathrm{tr} : B \to \mathbb{C}$.

The trivial quantum graph $TM_N = (B, \mathrm{tr}, A)$ is determined by the quantum adjacency matrix $A(x) = x$.

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Lemma

The quantum Cuntz-Krieger algebra C^*-algebra $\mathbb{F}\mathcal{O}(TM_N)$ is the universal C^*-algebra with generators S_{ij} for $1 \leq i, j \leq N$ satisfying the relations

$$\sum_{kl} S_{ik} S_{lk}^* S_{lj} = S_{ij}$$

$$\sum_{k} S_{ki}^* S_{kj} = \sum_{k} S_{ik} S_{jk}^*$$

for all i, j.

If we set $S = (S_{ij}) \in M_N(\mathbb{F}\mathcal{O}(TM_N))$ then the relations above read

$$SS^*S = S, \qquad S^*S = SS^*.$$

That is, we may say that $\mathbb{F}\mathcal{O}(TM_N)$ is the universal C^*-algebra generated by a normal $N \times N$-matrix partial isometry.

Example: Trivial quantum graphs

It is easy to check that $\mathbb{F}\mathcal{O}(TM_N)$ maps onto Brown's algebra U_N^{nc}, the universal C^*-algebra generated by the entries of a unitary $N \times N$-matrix $u = (u_{ij})$, by sending S_{ij} to u_{ij}.

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We may also map $\mathbb{F}\mathcal{O}(TM_N)$ onto the non-unital free product $\mathbb{C} * \cdots * \mathbb{C}$ of N copies of $\mathbb{C}$, by sending S_{ij} to $\delta_{ij} 1_i$, where 1_i denotes the unit element in the i-th copy of $\mathbb{C}$.

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This shows that the algebra $\mathbb{F}\mathcal{O}(TM_N)$ is not unital for $N > 1$.

Conclusion

The algebra $\mathbb{F}\mathcal{O}(TM_N)$ is neither unital, nuclear, nor simple.

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Theorem

Let TM_N be the trivial quantum graph as before. Then there exists a $*$-isomorphism

$$M_N(\mathbb{F}\mathcal{O}(TM_N)^+) \cong M_N(\mathbb{C}) *_1 (C(S^1) \oplus \mathbb{C}),$$

and the quantum Cuntz-Krieger algebra $\mathbb{F}\mathcal{O}(TM_N)$ is KK-equivalent to $C(S^1)$ for all $N \in \mathbb{N}$. In particular

$$K_0(\mathbb{F}\mathcal{O}(TM_N)) = \mathbb{Z},$$
$$K_1(\mathbb{F}\mathcal{O}(TM_N)) = \mathbb{Z}.$$

Here $*_1$ denotes the unital free product and $\mathbb{F}\mathcal{O}(TM_N)^+$ is the minimal unitarization of $\mathbb{F}\mathcal{O}(TM_N)$.

If we write $S = (S_{ij})$ for the matrix of generators of $\mathbb{F}\mathcal{O}(TM_N)$, then

- the generator of $K_0(\mathbb{F}\mathcal{O}(TM_N)$ is represented by the projection $S^*S \in M_N(\mathbb{F}\mathcal{O}(TM_N))$,
- the generator of $K_1(\mathbb{F}\mathcal{O}(TM_N)$ is represented by the unitary $S - (1 - S^*S) \in M_N(\mathbb{F}\mathcal{O}(TM_N)^+)$.

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Let $B = M_N(\mathbb{C})$ and $\mathrm{tr} : B \to \mathbb{C}$ the normalised trace.

The complete quantum graph $K(M_N(\mathbb{C}), \mathrm{tr})$ on B is determined by the quantum adjacency matrix $A(x) = N^2 \,\mathrm{tr}(x)$.

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Lemma

The quantum Cuntz-Krieger algebra $\mathbb{F}\mathcal{O}(K(M_N(\mathbb{C}), \mathrm{tr}))$ is the universal C^*-algebra with generators S_{ij} for $1 \leq i, j \leq N$ satisfying the relations

$$\sum_{kl} S_{ik} S_{lk}^* S_{lj} = S_{ij}$$

$$\sum_{r} S_{ri}^* S_{rj} = \delta_{ij} N \sum_{rs} S_{rs} S_{rs}^*$$

for all i, j.

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Lemma

Let $\mathbb{F}\mathcal{O}(K(B,\mathrm{tr}))$ *be as above. Then there exists a surjective* $*$*-homomorphism* $\phi : \mathbb{F}\mathcal{O}(K(B,\mathrm{tr})) \to \mathcal{O}_{N^2}$ *such that*

$$\phi(S_{ij}) = \frac{1}{N^{1/2}} s_{ij}$$

for all i, j*, where* s_{ij} *are standard generators of the Cuntz algebra* $\mathcal{O}_{N^2}$*.*

Proof.

We check

$$\sum_{rs} \phi(S_{ir})\phi(S_{sr})^* \phi(S_{sj}) = \sum_{rs} \frac{1}{N^{3/2}} s_{ir}(s_{sr})^* s_{sj} = \frac{1}{N^{1/2}} s_{ij} = \phi(S_{ij}),$$

$$\begin{aligned} \sum_{r} \phi(S_{ri})^* \phi(S_{rj}) = \sum_{r} \frac{1}{N}(s_{ri})^* s_{rj} = \delta_{ij} &= \delta_{ij} \sum_{kl} s_{kl}(s_{kl})^* \\ &= \delta_{ij} N \sum_{kl} \phi(S_{kl})\phi(S_{kl})^* \end{aligned}$$

as required. □

Example: Complete quantum graphs

Remark

This shows in particular that the canonical linear map $S : B \to \mathbb{F}\mathcal{O}(K(B, \mathrm{tr}))$ *is injective. This is not always the case for general quantum Cuntz-Krieger algebras.*

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This shows in particular that the canonical linear map $S : B \to \mathbb{F}\mathcal{O}(K(B, \mathrm{tr}))$ *is injective. This is not always the case for general quantum Cuntz-Krieger algebras.*

Our main structure result regarding $\mathbb{F}\mathcal{O}(K(B, \mathrm{tr}))$ can be stated as follows.

Theorem

The map $\phi : \mathbb{F}\mathcal{O}(K(M_N(\mathbb{C}), \mathrm{tr})) \to \mathcal{O}_{N^2}$ *is an isomorphism.*

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Theorem

The map $\phi : \mathbb{F}\mathcal{O}(K(M_N(\mathbb{C}), \mathrm{tr})) \to \mathcal{O}_{N^2}$ *is an isomorphism.*

In fact, we can prove the following stronger result:

Theorem

Let B *be an* n*-dimensional* C^**-algebra and let* $\psi : B \to \mathbb{C}$ *be a* δ*-form satisfying* $\delta^2 \in \mathbb{N}$*. Then* $\mathbb{F}\mathcal{O}(K(B, \psi)) \cong \mathcal{O}_n$*.*

Quantum symmetries

Quantum symmetries

A magic unitary $N \times N$-matrix is a matrix $u = (u_{ij})$ such that

$$u_{ij} = u_{ij} = u_{ij}^*$$

and

$$\sum_{k=1}^{n} u_{kj} = 1, \qquad \sum_{k=1}^{n} u_{ik} = 1$$

That is, all entries of u are projections and all row and columns sum to the identity.

Example

A magic unitary $u \in M_n(\mathbb{C})$ is the same thing as a permutation matrix.

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Definition (Wang 1998)

The quantum permutation group S_n^+ is the universal C^*-algebra $C(S_n^+)$ generated by the entries of a magic unitary $n \times n$-matrix $u = (u_{ij})$.

Quantum symmetries

If $E = (E^0, E^1)$ is a simple finite graph then the automorphism group $\mathrm{Aut}(E)$ consists of all bijections of E^0 which preserves the adjacency relation in E.

If $|E^0| = N$ and $A \in M_N(\mathbb{Z})$ is the adjacency matrix of E, then this can be expressed as

$$\mathrm{Aut}(E) = \{\sigma \in S_N \mid \sigma A = A\sigma\} \subset S_N,$$

where one views elements of the symmetric group as permutation matrices.

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If $E = (E^0, E^1)$ is a simple finite graph then the automorphism group $\mathrm{Aut}(E)$ consists of all bijections of E^0 which preserves the adjacency relation in E.

If $|E^0| = N$ and $A \in M_N(\mathbb{Z})$ is the adjacency matrix of E, then this can be expressed as

$$\mathrm{Aut}(E) = \{\sigma \in S_N \mid \sigma A = A\sigma\} \subset S_N,$$

where one views elements of the symmetric group as permutation matrices.

Definition (Banica)

The quantum automorphism group $G^+(E)$ of the graph E is the C^*-algebra

$$C(G^+(E)) = C(S_N^+)/\langle uA = Au\rangle,$$

where $u = (u_{ij}) \in M_N(C(S_N^+))$ denotes the defining magic unitary matrix.

This yields a quantum subgroup of S_N^+, which contains the classical automorphism group $\mathrm{Aut}(E)$ as a quantum subgroup.

Quantum symmetries

Let $\mathcal{G} = (B, \psi, A)$ be a quantum graph.

We say that an action $\beta : B \to B \otimes C(G)$ of a compact quantum group G is

- ψ-preserving if $(\mathrm{id} \otimes \psi)\beta(x) = \beta(x)1$ for all $x \in B$,
- compatible with $A : B \to B$ if $\beta \circ A = (A \otimes \mathrm{id}) \circ \beta$.

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Definition

Let $\mathcal{G} = (B, \psi, A)$ be a quantum graph. The quantum automorphism group $G^+(\mathcal{G})$ of $\mathcal{G}$ is the universal compact quantum group equipped with a ψ-preserving action $\beta : B \to B \otimes C(G^+(\mathcal{G}))$ which is compatible with the quantum adjacency matrix A.

Quantum symmetries

That is, the quantum automorphism group $G^+(\mathcal{G})$ has the following universal property.

If G is a compact quantum group and $\gamma : B \to B \otimes C(G)$ an action of G which preserves ψ and is compatible with A, then there exists a unique morphism $\pi : C(G^+(\mathcal{G})) \to C(G)$ such that

$$\begin{array}{ccc} B & \xrightarrow{\ \beta\ } & B \otimes C(G^+(\mathcal{G})) \\ & \searrow^{\gamma} & \downarrow{\scriptstyle \mathrm{id} \otimes \pi} \\ & & B \otimes C(G) \end{array}$$

commutes.

Quantum symmetries

Theorem

Let $\mathcal{G} = (B, \psi, A)$ be a quantum graph. Then the canonical action $\beta : B \to B \otimes C(G^+(\mathcal{G}))$ of the quantum automorphism group of $\mathcal{G}$ induces an action $\hat{\beta} : \mathbb{F}\mathcal{O}(\mathcal{G}) \to \mathbb{F}\mathcal{O}(\mathcal{G}) \otimes C(G^+(\mathcal{G}))$ such that

$$\hat{\beta}(S(b)) = (S \otimes \mathrm{id})\beta(b)$$

for all $b \in B$.

This generalises to actions of the linking algebras of quantum isomorphic quantum graphs.

The latter is key to the proof of our main theorem on quantum Cuntz-Krieger algebras associated with complete quantum graphs.